

# A sub-constant improvement in approximating the positive semidefinite Grothendieck problem

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## Abstract

Semidefinite relaxations are a powerful tool for approximately solving combinatorial optimization problems such as MAX-CUT and the Grothendieck problem. By exploiting a bounded rank property of extreme points in the semidefinite cone, we make a sub-constant improvement in the approximation ratio of one such problem. Precisely, we describe a polynomial-time algorithm for the positive semidefinite Grothendieck problem – based on rounding from the standard relaxation – that achieves a ratio of  $2/\pi + \Theta(1/\sqrt{n})$ , whereas the previous best is  $2/\pi + \Theta(1/n)$ . We further show a corresponding integrality gap of  $2/\pi + \tilde{O}(1/n^{1/3})$ .

## 1 Introduction

Given a positive semidefinite (PSD) matrix  $A \in \mathbb{R}^{n \times n}$ , the positive semidefinite Grothendieck problem is

$$\max_{x \in \{-1,1\}^n} \text{QP}_A(x) \stackrel{\text{def}}{=} x^\top A x. \quad (1)$$

The problem is NP-hard; it is easy to see that MAX-CUT arises as the special case when  $A$  is a graph Laplacian. Elsewhere, the problem has applications ranging from graph partitioning (Alon and Naor, 2006) to kernel clustering (Khot and Naor, 2008, 2010). See Pisier (2012) for a broad survey.

The polynomial-time algorithm that achieves the asymptotically best known approximation ratio for this problem – the constant  $2/\pi \approx 0.637$  – is essentially the same as that described by Goemans and Williamson (1995) for MAX-CUT: the problem (1) is relaxed to a convex semidefinite program (SDP) that is equivalent to

$$\begin{aligned} & \max_{X \in \mathbb{R}^{n \times n}} && \text{tr}(X^\top A X) \\ & \text{subject to} && \|X_i\|_2^2 = 1 \end{aligned} \quad (2)$$

where  $\{X_i\}$  are the rows of  $X$ . We can think of each  $X_i$  as comprising a relaxation of the binary variable  $x_i$  to an  $n$ -dimensional real unit vector. This convex relaxation is solved to arbitrary accuracy and its solution randomly rounded to a discrete one for (1).<sup>1</sup>

There is evidence suggesting that this approximation is asymptotically optimal. In particular, Alon and Naor (2006) exhibit a random problem instance  $A$  whose asymptotic integrality gap is  $2/\pi$ . For the SDP relaxation approach, the integrality gap bounds the approximation ratio from above. In general, Khot and Naor (2008) show that if the unique games conjecture holds then no polynomial time algorithm can exceed a  $2/\pi$  ratio guarantee.

However, we can still push further against this barrier. To do so, we look to approximation ratios that, for any *fixed*  $n$ , exceed an asymptotic limit of  $2/\pi$ , and to favor those algorithms whose ratio decays *more slowly* to the asymptotic limit. Such an algorithm is said to provide a *sub-constant* improvement to the  $2/\pi$ -approximation. It is the best kind of improvement possible that avoids confrontation with asymptotic hardness barriers.

The initial proof that the SDP rounding procedure used in Goemans and Williamson (1995) for MAX-CUT can be repurposed for a  $2/\pi$  approximation to the PSD Grothendieck problem is due to Nesterov (1998). More recently, Briët et al. (2010) showed, by a careful analysis, that the procedure in fact achieves an approximation ratio of  $2/\pi + \Theta(1/n)$ . Intuitively, as the problem instance grows, the dimension of the relaxed variables  $\{X_i\}$  grows with it, and the expected gain of rounding (over  $2/\pi$ ) decreases inversely with the relaxed dimension.

The key to our improvement is twofold. First, the expected gain is larger when the relaxed variables  $\{X_i\}$  all lie in a low-dimensional subspace of  $\mathbb{R}^n$ . This leads us to seek a polynomial-time dimensionality reduction that improves the expected gain more than it decreases the SDP objective value. Second, by controlling the matrix rank of optimal solutions to semidefinite programs, we can actually obtain an immediate such dimensionality reduction – down to below  $\sqrt{2n}$  – at *entirely no cost in objective*. To our knowledge, this is the first use of such an essential property of SDP extreme points in the context of approximation by SDP relaxation. Lastly, our analysis can be sharpened for problem instances in which  $A$  itself is low rank. In these cases, we show an approximation ratio of  $2/\pi + \Theta(1/\text{rank}(A))$ . In particular, this implies a characterization of problem instances – those with  $A$  of constant rank – for which the algorithm we present achieves a *constant* improvement.

## 1.1 Formal setup and main result

**Notation** We write  $\mathbb{S}_k$  for the set of symmetric  $k \times k$  real matrices and  $\mathcal{S}^k$  for the unit sphere  $\{x \in \mathbb{R}^k : \|x\|_2 = 1\}$ . All vectors are columns unless stated otherwise. If  $X$  is a matrix, then  $X_i \in \mathbb{R}^{1 \times k}$  is its  $i$ 'th row.

We are interested in approximately solving the positive semidefinite Grothendieck problem (1) by rounding an optimal solution of the relaxed problem (2). As stated, (2) is not convex,

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<sup>1</sup>Though the algorithms are the same, the MAX-CUT approximation constant (about 0.878) exceeds  $2/\pi$  by an analysis that exploits all-positive edge weights and the graph Laplacian structure of  $A$ .

but it does correspond exactly to a (convex) semidefinite program through the change of variables  $S = XX^\top$ :

$$\begin{aligned} & \max_{S \in \mathbb{S}_n} \quad \text{SDP}_A(S) \stackrel{\text{def}}{=} \text{tr}(AS) \\ & \text{subject to} \quad S \succeq 0, \quad \text{diag}(S) = \mathbf{1} \end{aligned} \quad (3)$$

Note that if the rank of some feasible  $S$  equals  $k$ , then the corresponding  $X$  feasible for (2) has rows that are effectively  $k$ -dimensional.

Because  $\text{SDP}_A$  is a convex program, we can obtain an optimal solution  $S^*$  of  $\text{SDP}_A$  within a desired precision  $\epsilon$  in time polynomial in  $n$  and  $\log(1/\epsilon)$ . From an optimal SDP point  $S^*$ , we can obtain a feasible point  $\hat{x} \in \{-1, 1\}^n$  for  $\text{QP}_A$  by the following randomized rounding procedure: factor  $S$  into  $XX^\top$ , sample a random vector  $g$  from the unit sphere  $\mathcal{S}^k$ , and output  $\hat{x} = \text{sign}(Xg)$ .

This randomized rounding is analyzed independently by Goemans and Williamson (1995) and by Nesterov (1998). Both show that the approximation ratio is bounded above and below as follows:

$$\frac{2}{\pi} \leq \frac{\mathbb{E}[\text{QP}_A(\hat{x})]}{\text{SDP}_A(S^*)} \leq 1 \quad (4)$$

and, as mentioned above, this is the asymptotically optimal approximation ratio of any polynomial-time algorithm, provided that the unique games conjecture holds.

Adapting the rounding analysis of Briët et al. (2010) and controlling the rank of SDP solutions, we obtain in this paper an approximation ratio of

$$\frac{2}{\pi} + \frac{1}{\pi\sqrt{2n}} + o\left(\frac{1}{\sqrt{n}}\right) = \frac{2}{\pi} + \Theta\left(\frac{1}{\sqrt{n}}\right). \quad (5)$$

Section 2 shows that solutions of  $\text{SDP}_A$  with low rank – bounded above by  $\sqrt{2n}$  – always exist, and describes a polynomial-time algorithm for finding them. Section 3 shows the approximation ratio achieved by the randomized rounding algorithm applied to a  $k$ -rank solution of  $\text{SDP}_A$  for a known  $k \leq n$ . Combining these results via  $k = \sqrt{2n}$  yields the main result (5). Finally, Section 4 adapts the analysis of Alon and Naor (2006) to show a corresponding upper bound of the integrality gap – and hence the best approximation guarantee possible via the SDP rounding approach – is at most  $2/\pi + \tilde{O}(1/n^{1/3})$ .

In addition to the main result, Section 2.1 adapts the rank reduction algorithm in 2.1 to further improve the approximation ratio whenever  $A$  has rank  $o(\sqrt{n})$ .

## 2 SDP solution rank

Considering only the constraint count of a semidefinite program, while ignoring its objective altogether, Barvinok (1995) and Pataki (1998) argue geometrically that SDP solutions have bounded rank:

**Theorem 2.1** (Barvinok (1995); Pataki (1998)). *Any semidefinite program of  $m$  linear constraints has an optimal solution  $S^*$  such that  $t(\text{rank}(S^*)) \leq m$ , where  $t(k) = k(k+1)/2$  is the  $k$ 'th triangular number.*

Since  $\text{SDP}_A$  has only  $n$  constraints – those of the form  $S_{ii} = 1$  – it follows that it has an optimal solution whose rank does not exceed roughly  $\sqrt{2n}$ .

Afakih and Wolkowicz (1998) give a concrete algorithm for finding the low-dimensional Euclidian embeddings shown to exist in the proof of Barvinok (1995). The algorithm is essentially a constructive version of the existence proof concurrently given by Pataki (1998). By simplifying their key ideas and translating them to the problem of rank-reducing solutions of  $\text{SDP}_A$ , we obtain Algorithm 1, which reduces the rank of an SDP solution  $S$  without changing its objective value nor affecting feasibility. The algorithm proceeds by solving a homogeneous linear system that is underdetermined whenever  $\text{rank}(S)$  is sufficiently large.

**Input** : SDP solution  $S \in \mathbb{S}_n$  of rank  $k$ , with  $t(k) > n + 1$ .

**Output**: SDP solution  $S'$  of rank  $k'$ , with  $t(k') \leq n + 1$ .

Note that  $t(k) = k(k+1)/2$  is the dimension of  $\mathbb{S}_k$ .

Factor  $S = XX^\top$  with  $X \in \mathbb{R}^{n \times k}$ .

Solve  $n + 1$  homogeneous linear equations in  $t(k)$  variables  $Y \in \mathbb{S}_n$ :

$$\text{tr}(X_i^\top X_i Y) = 0 \text{ for each of } X\text{'s } n \text{ row vectors } X_i \in \mathbb{R}^{1 \times k}$$

$$\text{tr}(X^\top A X Y) = 0$$

Negate and scale  $Y \neq 0$  if needed, so that its largest eigenvalue  $\lambda_{\max} = 1$ .

Set  $U \leftarrow I_k - Y$  and  $S' \leftarrow XU X^\top$ .

**Algorithm 1:** Rank-reduction of an  $\text{SDP}_A$  solution

To see that Algorithm 1 delivers on its promises, observe that  $\text{rank}(S') \leq \text{rank}(U) < \text{rank}(S) = k$  because  $\det(U) = \det(Y - \lambda_{\max} I_k) = 0$  for the eigenvalue  $\lambda_{\max} = 1$  of  $Y$ . We can check that  $U \succeq 0$  and therefore  $S' \succeq 0$ . Because we found  $Y$  satisfying the linear system, we can also check that as far the constraints and objective of  $\text{SDP}_A$  are concerned,  $S'$  is no worse than  $S$ . The resulting objective value is

$$\begin{aligned} \text{tr}(AS') &= \text{tr}(AXUX^\top) = \text{tr}(X^\top AXU) = \text{tr}(X^\top AX(I_k - Y)) \\ &= \text{tr}(X^\top AX I_k) = \text{tr}(AXX^\top) = \text{tr}(AS). \end{aligned} \tag{6}$$

Similarly, the new solution remains feasible:

$$S'_{ii} = X_i U X_i^\top = \text{tr}(X_i^\top X_i U) = \text{tr}(X_i^\top X_i I_k) = S_{ii} = 1. \tag{7}$$

## 2.1 Low rank problem instances

Further rank-reduction is possible for problem instances with additional structure. In this section we show that, when  $A$  is low rank, it is possible to modify Algorithm 1 so that it reduces solution rank to the rank of  $A$ .

To exploit the rank of  $A$ , we replace the linear homogeneous equations in Algorithm 1 with the semidefinite program,

$$Y \succeq 0, \operatorname{tr}(X^\top A X Y) = 0, Y \neq 0, \quad (8)$$

and claim that it is feasible whenever  $k > \operatorname{rank}(A)$ . To see this, diagonalize  $X^\top A X = Q \operatorname{diag}(\lambda) Q^\top$  with orthonormal eigenvectors  $Q \in \mathbb{R}^{n \times k}$  and eigenvalues  $\lambda \in \mathbb{R}^k$ . Since  $k > \operatorname{rank}(A)$ , there exists  $i$  such that  $\lambda_i = 0$ . If we assign the non-zero vector  $\lambda' \in \mathbb{R}^k$  as

$$\lambda'_j \leftarrow \mathbb{I}[\lambda_j = 0], \quad j = 1, \dots, k, \quad (9)$$

then  $Y = Q \operatorname{diag}(\lambda') Q^\top$  satisfies (8).

After solving for  $Y$ , the last step of Algorithm 1 computes the rank-reduced solution  $S'$ . We can follow (6) to check that  $S'$  gives us the same objective value. For feasibility, we have

$$S'_{ii} = X_i U X_i^\top = X_i (I_k - Y) X_i^\top = S_{ii} - X_i Y X_i^\top = 1 - X_i Y X_i^\top.$$

Recall  $\lambda_{\max}(Y) = 1$ ,  $Y \succeq 0$ , and  $\|X_i\|_2 = 1$ . Therefore,  $0 \leq S'_{ii} \leq 1$ .<sup>2</sup>

### 3 Rounding from low rank

The following lemma states that the approximation ratio due to randomized rounding is better when rounding from lower-rank SDP solutions. The statement is a simple consequence of Lemma 1 of Briët et al. (2010), which makes important use of the results of Schoenberg (1942) together with Grothendieck's identity.

**Lemma 3.1.** *Fix a weight matrix  $A \succeq 0$  and  $X \in \mathbb{R}^{n \times k}$  with  $X_i \in \mathcal{S}^k$ , the unit sphere. Let  $g$  be a random vector from  $\mathcal{S}^k$  and*

$$\gamma(k) \stackrel{\text{def}}{=} \frac{2}{k} \left( \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \right)^2 = 1 - \Theta\left(\frac{1}{k}\right). \quad (10)$$

*Then the expected approximation ratio obtained by randomized rounding*

$$R(k) \stackrel{\text{def}}{=} \frac{\mathbb{E}_g[\operatorname{QP}_A(\operatorname{sign}(Xg))]}{\operatorname{SDP}_A(XX^\top)} \quad (11)$$

*is at least*

$$\frac{2}{\pi \gamma(k)} = \frac{2}{\pi} \left( 1 + \frac{1}{2k} + o\left(\frac{1}{k}\right) \right). \quad (12)$$

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<sup>2</sup>Although the constraint  $S_{ii} = 1$  appears in the formal problem setup, the constraint  $S_{ii} \leq 1$  is equivalent for the PSD Grothendieck problem due to having  $A \succeq 0$ .

*Proof.* Grothendieck's identity states that, for  $u, v \in \mathbb{R}^k$  and  $g$  drawn uniformly from the unit sphere  $\mathcal{S}^k$ ,

$$\mathbb{E}_g [\text{sign}(u^\top g) \text{sign}(v^\top g)] = \frac{2}{\pi} \arcsin(u^\top v). \quad (13)$$

Let  $Y = f(XX^\top) \in \mathbb{R}^{n \times n}$  be the elementwise application of the scalar function

$$f(t) = \frac{2}{\pi} \left( \arcsin(t) - \frac{t}{\gamma(k)} \right). \quad (14)$$

Lemma 1 in Briët et al. (2010) shows that  $f(t)$  is a function of the *positive type* on  $\mathcal{S}^k$ , which by definition means that  $Y \succeq 0$  provided  $X_i \in \mathcal{S}^k$  for all  $i$ . Their result is based on (a) computing inner products between orthogonal Jacobi polynomials, together with (b) the characterization due to Schoenberg (1942) of positive definite functions on  $\mathcal{S}^k$  in terms of Jacobi polynomials.

We have that  $\text{tr}(AY) \geq 0$ . Rearranging terms and applying Grothendieck's identity:

$$0 \leq \text{tr}(AY) = \text{tr} \left( A \frac{2}{\pi} \left( \arcsin(XX^\top) - \frac{XX^\top}{\gamma(k)} \right) \right) \quad (15)$$

$$\iff \text{tr} \left( A \frac{2}{\pi} \arcsin(XX^\top) \right) \geq \frac{2}{\pi \gamma(k)} \text{tr}(AXX^\top) \quad (16)$$

$$\iff \mathbb{E}_g[\text{QP}_A(\text{sign}(Xg))] \geq \frac{2}{\pi \gamma(k)} \text{SDP}_A(XX^\top), \quad (17)$$

which proves the claim.  $\square$

## 4 Integrality gap

How much further could we hope to improve the additive sub-constant term in the ratio between rounded and relaxed solutions? This section bounds the answer by providing an integrality gap of  $2/\pi + \tilde{O}(1/n^{1/3})$ .

To establish the gap, we set out to construct, for every  $n$ , a matrix  $A \in \mathbb{R}^{n \times n}$  so that  $\frac{\text{QP}_A(x^*)}{\text{SDP}_A(S^*)} \leq 2/\pi + \tilde{O}(1/n^\alpha)$ . The particular construction we consider achieves  $\alpha = 1/3$ . We first outline and reproduce some results from Section 5.2 of Alon and Naor (2006), and then expand them to analyze the sub-constant rates.

The authors' original construction uses  $n$  random unit vectors  $v_i \in \mathcal{S}^p$  for  $i = 1, \dots, n$  and takes  $A_{ij} = \frac{1}{n} v_i^\top v_j$ . If we set  $S_{ij} = A_{ij}$  then

$$\text{SDP}_A(S^*) \geq \text{SDP}_A(S) = \text{tr}(AS) = \frac{1}{n^2} \sum_{ij} (v_i^\top v_j)^2 \rightarrow 1/p, \quad (18)$$

where  $1/p$  arises as the average inner product between random vectors on  $\mathcal{S}^p$ .

Under the QP, for any  $x \in \{-1, 1\}^n$ , we have

$$\text{QP}_A(x) = \sum_{i,j} A_{ij} x_i x_j = \left\| \frac{1}{n} \sum_{i=1}^n v_i x_i \right\|^2. \quad (19)$$

Take  $x^* \in \arg \max_x \text{QP}_A(x)$  and let  $c$  be the unit vector the direction of  $\sum_{i=1}^n x_i^* v_i$ . It is optimal to accumulate in the correct direction  $c$ , so  $x_i^* = \text{sign}(v_i^\top c)$  and hence

$$\text{QP}_A(x^*) = \left( \frac{1}{n} \sum_{i=1}^n x_i^* v_i^\top c \right)^2 = \left( \frac{1}{n} \sum_{i=1}^n |v_i^\top c| \right)^2 \rightarrow (\mathbb{E}[|v^\top c|])^2. \quad (20)$$

Alon and Naor (2006) computed this expectation; it is easy to verify that the sub-constant term  $\Theta(1/p)$  appears therein as follows:

$$\mathbb{E}[|v^\top c|] = \left( \sqrt{\frac{2}{\pi}} + \Theta\left(\frac{1}{p}\right) \right) \frac{1}{\sqrt{p}}. \quad (21)$$

We would now like to maximize an  $n$ -sample estimate of (21) over the sphere. The original analysis does this by replacing maximization over the sphere with the same over a corresponding  $\epsilon$ -net:

$$\text{QP}_A(x^*) = \left( \max_{d \in \mathcal{S}^p} \sum_{i=1}^n \frac{1}{n} |v_i^\top d| \right)^2 = \left( \left( \max_{d \in \epsilon\text{-net}(\mathcal{S}^p)} \sum_{i=1}^n \frac{1}{n} |v_i^\top d| \right) + O(\epsilon) \right)^2. \quad (22)$$

Now,  $n$  needs to be big enough so the variance of the  $n$ -sample estimator,

$$\text{var} \left[ \sum_{i=1}^n \frac{1}{n} |v_i^\top d| \right] = \frac{1}{n} \text{var}[|v_i^\top d|] = O\left(\frac{1}{np}\right), \quad (23)$$

is small enough to safely maximize over an  $\epsilon$ -net of size  $O(\frac{1}{\epsilon^p})$ . The integrality gap question then reduces to the question of how big  $n$  should be.

To handle the max, we observe that  $\sum_{i=1}^n \frac{1}{n} |v_i^\top d|$  is sub-Gaussian with parameter  $O(1/\sqrt{np})$ , so it enjoys the following bound: If  $X_i \sim \mathcal{N}(0, \sigma^2)$  (or if  $X_i$  is sub-Gaussian with parameter  $\sigma$ ) are i.i.d across  $i = 1, \dots, m$ , then  $\mathbb{E}[\max_i(X_i)] \leq \sigma \sqrt{2 \log(m)}$ .

Now we proceed to bound  $\text{QP}_A(x^*)$  from above:

$$\text{QP}_A(x^*) = \left( \left( \max_{d \in \epsilon\text{-net}(\mathcal{S}^p)} \sum_{i=1}^n \frac{1}{n} |v_i^\top d| \right) + O(\epsilon) \right)^2 \quad (24)$$

$$\leq \left( \left( \sqrt{\frac{2}{\pi}} + \Theta\left(\frac{1}{p}\right) \right) \frac{1}{\sqrt{p}} + \sqrt{\frac{2 \log(\frac{1}{\epsilon^p})}{np}} + O(\epsilon) \right)^2. \quad (25)$$

Pick a small enough  $\epsilon$  so that the  $O(\epsilon)$  term may be ignored. This can be done because the second additive term only grows as  $\log(1/\epsilon)$ , so we can pick  $\epsilon = o(1/(p\sqrt{p}))$  to enforce that the first additive term is dominant. Multiply both sides of (24) by the inequality  $1/\text{SDP}_A(S^*) \leq p$  shown in (18). This yields:

$$\frac{\text{QP}_A(x^*)}{\text{SDP}_A(S^*)} \leq \left( \left( \sqrt{\frac{2}{\pi}} + \Theta\left(\frac{1}{p}\right) \right) + \sqrt{\frac{2p \log(\frac{1}{\epsilon})}{n}} \right)^2. \quad (26)$$

In order to balance the two sub-constant additive terms, we can set  $n = p^3$ . This construction has integrality gap less than  $2/\pi + \tilde{O}(1/n^{1/3})$ .

## 5 Concluding remarks

We demonstrated a sub-constant improvement in approximating the PSD Grothendieck problem. Although the improvement disappears asymptotically, it decays slowly via an additive term whose constant factors we have made explicit. Two of the three main ingredients of this result are obtained by adapting existing analyses to explicitly account for effective relaxed dimension in the “first order” sub-constant additive term. The remaining ingredient comes from exploiting the spectral sparsity of extreme points in the SDP cone, an analysis tool of independent interest. With the same tool set, we further characterized a class of problem instances for which the new approximation ratio enjoys an additional – even constant – advantage.

An immediate direction for future work is to ask whether the sub-constant improvement described here has downstream implications for other approximation algorithms. Another is whether the result can be improved, or conversely whether the integrality gap is actually smaller than shown in Section 4. A more general question is whether this same set of tools can be applied to other SDP relaxation-based algorithms in order to improve their approximation ratio – by an additive sub-constant term or otherwise – with immediate candidates being MAX-CUT,  $k$ -coloring, and kernel clustering.

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